

An uncertainty principle and lower bounds for the Dirichlet Laplacian on graphs

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This paper is dedicated to W. Kirsch and B. Simon as part of the celebration of their recent birthdays. We are grateful for their inspiration.

Abstract

We prove a quantitative uncertainty principle at low energies for the Laplacian on fairly general weighted graphs with a uniform explicit control of the constants in terms of geometric quantities. A major step consists in establishing lower bounds for Dirichlet eigenvalues in terms of the geometry.

1 Introduction

It is a phenomenon of general interest that low energy states of Laplacians are extended in some sense. Several closely related concepts deal with that fact. One of them is *unique continuation* for subsolutions of elliptic equations. We refer to [1, 2, 4, 23, 25] for a small selection of the long list of contributions and remark that there was renewed interest in quantitative versions due to the importance of such results for random Schrödinger operators, as seen in [9]; see also [8, 33, 34, 38, 40] and the literature quoted there for more recent results. In its original form, unique continuation means that such subsolutions cannot vanish to infinite order. This is true in a variety of continuum contexts and certainly not true for graph Laplacians. In fact, discrete Laplacians even allow for eigenfunctions with compact support. For the special case of a tight binding model associated with the Penrose tiling the occurrence of this effect has been known since quite some time as witnessed for example in the physics literature [3, 18, 36, 35]. This phenomenon

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is also interesting from a mathematical point of view, see [15, 32, 44]. For certain planar lattices, however, unique continuation holds due to curvature conditions as first shown [31] and later generalized in [27].

For the above mentioned applications to random Schrödinger operators quite a different point of view is important. Namely, states are required to be extended in the sense that the norm of restrictions to subsets remains relevant, provided the subset one restricts the function to is spread out in space. Many of the references above deal with that kind of *uncertainty principle* and establish such kind of lower bounds provided the function in question is an eigenfunction of a Schrödinger operator, or more generally in the range of the spectral projection of a Schrödinger operator onto a small interval of the energy axis. In view of the above mentioned phenomenon of compactly supported eigenfunctions, one cannot hope for an analogous result in the discrete case. However, as we will show in the present paper, uncertainty principles hold for low energy states of graph Laplacians and the results allow for a uniform estimate for large classes of graphs, with an explicit control of constants phrased in terms of geometric properties. While our results are pretty general, we stress the fact that they provide new insights even in the most simple cases, e.g. the usual euclidean lattices \mathbb{Z}^d . For this case, related results have been found in [16, 39]. For a more detailed comparison we refer to the discussion following our main Theorem 5.1.

Starting point of our method of proof is a spectral theoretic uncertainty principle, Theorem 1.1 from [10]. It deals with a semibounded selfadjoint operator H in some Hilbert space, a bounded nonnegative operator W , and phrases uncertainty or unique continuation in terms of the spectral projections $P_I = P_I(H)$ of H . It says that

$$P_I W P_I \geq \kappa P_I \tag{1}$$

provided there is $t > 0$ such that

$$\max I < \min \sigma(H + tW) =: \lambda_t \tag{2}$$

Actually, in this case an explicit lower bound on κ is easily established, viz

$$\kappa \geq \frac{\lambda_t - \max I}{t}.$$

For the application we have in mind, H is the Laplacian on a weighted graph X that obeys some mild assumptions and $W = 1_D$ is the indicator function of a subset that is spread out in X in the sense that for some $R > 0$

$$X \subset \bigcup_{p \in D} B_R(p).$$

This condition is also known as relative denseness of the set D in X . It is clear that in this case (1) amounts to

$$\|\phi\|^2 \leq \kappa^{-1} \|\phi 1_D\|^2 \text{ for all } \phi \in \text{Ran}(P_I)$$

meaning that we have a quantitative unique continuation result for linear combinations of eigenfunctions with eigenvalues in I and more general functions in the range of the corresponding spectral projection. Here κ depends on I and the optimal R that satisfies the above covering condition, see [10]. As explained above, such unique continuation estimates are somewhat astonishing in the graph setting since graph Laplacians can exhibit compactly supported eigenfunctions. However, our main result does not exclude such compactly supported eigenfunctions and applies to graphs where the latter occur. This is not a contradiction as our result only applies to energy intervals I concentrated near 0.

The idea of our method can be summarized as follows. Sending $t \rightarrow \infty$ in (2) with $W = 1_D$ we see that the maximal energy range for which (1) gives nontrivial results is determined by

$$\sup_{t>0} \min \sigma(H + t1_D) \stackrel{!}{=} \min \sigma(H + \infty 1_D),$$

where the non-densely defined form sum

$$H + \infty 1_D =: H_\Omega$$

is the Dirichlet Laplacian (in a suitable sense) on $\Omega := X \setminus D$.

Our first task is therefore to get lower bounds for this H_Ω in terms of geometric quantities of the underlying graph and the sets Ω and D , respectively. This is discussed in Section 3. Theorem 3.10 shows

$$H_\Omega \geq \frac{1}{R \cdot \sup\{\text{vol}(B_R(p)) \mid p \in D\}}$$

where $R = \text{Inr}(\Omega)$ is the inradius of Ω , see Section 2 below for the definition of volume in our weighted graph setting. This bound is a generalization of a well-known bound for finite graphs to infinite geometries under some mild assumptions on the weighted graph. In our proof of the theorem, we reduce the infinite graph to a disjoint union of finite graphs and this is a crucial step in our approach. It is achieved via a Voronöi type decomposition. The existence of such a decomposition may be of interest in other contexts as well. To show this existence we need the rather careful analysis of basic features of the underlying geometry provided in Section 2. Note that the bound in the theorem is weaker than what is known in the euclidean case for \mathbb{R}^N , where the corresponding Dirichlet Laplacian is bounded below by $\text{const} R^{-2}$ for domains with nice enough boundary, see Theorem 1.5.8 in Davies [13]. In Section 3 we also give upper bounds on the ground state energy for graph

Laplacians that show that a lower bound like in the continuum will not be possible, even for very nice graphs.

In order to use (1) we will further need to control the convergence of $\min \sigma(H + t1_D)$ to $\min \sigma(H_\Omega)$. Luckily, we are in a discrete situation since the corresponding convergence would not be true in euclidean space. In our case, under the assumption that our Laplacian H is bounded, we get convergence in norm resolvent sense for the operators and with an explicit convergence rate of optimal order, as shown in Section 4 below.

It is then easy to put things together in Section 5 and obtain our main results, Theorem 5.1 and its corollary, giving a version of (1) for the case at hand with explicit control over κ in terms of the geometry.

In Section 6 we further discuss the case of combinatorial graphs and, in particular, compare our approach to lower bounds for H_Ω in Theorem 3.10 with the approach via Cheeger inequalities. Finally, in Section 7 we discuss how Theorem 3.10 can be extended to certain cases where a potential is added to H_Ω .

2 The set-up

We start by introducing our basic set-up within the context of weighted graphs, see [28] for a recent survey of this and related topics. A weighted graph (X, b, m) is given by

- a countable set X , finite or infinite;
- a symmetric weight function $b : X \times X \rightarrow [0, \infty)$ with $b(x, x) = 0$ for all $x \in X$ and $\sum_{y \in X} b(x, y) < \infty$ for all $x \in X$;
- a weight function $m : X \rightarrow (0, \infty)$.

Here m induces a measure on X through

$$\text{vol}(A) := m(A) := \sum_{x \in A} m(x).$$

Our basic Hilbert space will be

$$\ell^2(X, m) := \{f \in \mathbb{C}^X \mid \|f\|^2 = \sum_{x \in X} |f(x)|^2 m(x) < \infty\}.$$

The function b above should be thought of as a weight on the edges and it appears in the energy form of the Laplacian as well as in the distance we define on X . More precisely, we consider the nonnegative form

$$\mathcal{E}(f, g) := \frac{1}{2} \sum_{x, y \in X} b(x, y) (f(x) - f(y))(\overline{g(x)} - \overline{g(y)}).$$

We will always assume the boundedness condition

$$(B) \sup_{x \in X} \frac{1}{m(x)} \sum_{y \in X} b(x, y) =: \delta < \infty.$$

This condition is equivalent to boundedness of the form and consequently, the associated selfadjoint operator H ; more precisely, $\|H\| \leq 2\delta$. See [22], Thm. 9.3 and the literature cited there. The associated selfadjoint operator is known as *weighted Laplacian* and given by

$$(Hf)(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x) - f(y))$$

for $f \in \ell^2(X, m)$ and $x \in X$, as follows from [29], Theorem 5.

For us the relevant distance on X is given in the following way: an *edge* of the weighted graph (X, b, m) is a set $\{x, y\}$ with positive weight $b(x, y) > 0$. Denote by E the set of all edges. Clearly, that induces the structure of a combinatorial graph (X, E) . A *path* is a finite sequence of edges with nonempty intersections that can most easily be written as $\gamma = (x_0, x_1, \dots, x_k)$ where $b(x_j, x_{j+1}) > 0$ for all $j = 0, \dots, k-1$; if we want to specify the endpoints we say that γ is a path from x_0 to x_k . The *length* of such a path γ is given by

$$L(\gamma) := \sum_{j=0, \dots, k-1} \frac{1}{b(x_j, x_{j+1})}.$$

In particular the length of an edge $\{x, y\}$ is given by $\frac{1}{b(x, y)}$. To include trivial cases we also allow trivial paths (x, x) from x to x whose length is 0. We will throughout assume that our graph is *connected* in the sense that every pair of points is connected by a path. The *distance* between x and y is given by

$$d(x, y) := \inf\{L(\gamma) \mid \gamma \text{ a path from } x \text{ to } y\}.$$

Clearly, d is symmetric and satisfies the triangle inequality. Moreover, by the assumptions on b we have for any $x \in X$ the estimate

$$\sup_z b(x, z) \leq \sum_z b(x, z) < \infty$$

and this implies $d(x, y) \geq \frac{1}{\sup_z b(x, z)} > 0$ for any $y \in X$ with $y \neq x$. So, we see that d separates the points and hence we get that d is a metric. We denote by

$$U_r(x) := \{y \in X \mid d(x, y) < r\} \text{ and } B_r(x) := \{y \in X \mid d(x, y) \leq r\}$$

the open and closed balls of radius r , respectively. We note in passing that, while the graph (X, E) is connected, X is totally disconnected in the topological sense as it is discrete (by what we have just shown).

For our later considerations we will need the Heine-Borel property i.e. that closed balls in X are compact. As our space has discrete topology this is equivalent to the following finiteness condition:

(F) For any $x \in X$ and $r > 0$ the set $B_r(x)$ is finite.

We will also need that X is *geodesic* in the sense that the following condition holds:

(G) Between any $x, y \in X$ there exists a path $\gamma = (x_0, x_1, \dots, x_k)$ with $x_0 = x$, $x_k = y$ and $d(x, y) = L(\gamma)$.

It is easy to see that (F) implies (G) (compare also proof of part (d) of Proposition 2.1 below). In fact even the converse is true [24], see Remark 2.4 below as well.

By what we already mentioned in the introduction, the volume of balls will enter our results as one important quantity. In particular, we will need uniform bounds on the volumes of balls of fixed radius:

(V) For any $r \geq 0$ the inequality $\sup_{x \in X} m(B_r(x)) < \infty$ holds.

As $r = 0$ is possible, the previous condition clearly implies a uniform bound on m in the following form:

(M) $m_{\max} := \sup_{x \in X} m(x) < \infty$

Given (B), it turns out that (M) alone already implies (F), (G) and (V). In fact, (B) and (M) together can be seen to imply a rather homogeneous geometry. This is discussed next. The crucial point is that (B) and (M) together imply a uniform upper bound for b (and even for the *vertex degree* $\deg(x) = \sum_y b(x, y)$):

$$b_{\max} := \sup_{x, y \in X} b(x, y) \leq \sup_{x \in X} \sum_{y \in X} b(x, y) \leq \delta \cdot m_{\max}.$$

Proposition 2.1 (Homogeneity of the geometry). *Let (X, b, m) be as above, in particular connected and such that (B) and (M) hold. Then*

- (a) *For any path γ we have $L(\gamma) \geq b_{\max}^{-1} \# \gamma$, where $\#$ indicates the cardinality. In particular, X is uniform discrete; more precisely any two different points $x, y \in X$ have uniform distance at least $\frac{1}{b_{\max}}$.*
- (b) *(X, d) is locally compact and complete.*
- (c) *The condition (F) holds in a very uniform manner. More specifically,*

$$\#B_r(x) \leq (r \cdot \delta \cdot m_{\max})^{r \cdot b_{\max}} + 1$$

for any $r \geq 0$ and $x \in X$. In particular, (V) holds.

- (d) *(X, d) is geodesic, i.e. (G) holds.*

Proof. (a) As $b(x, y) \leq b_{max}$ for all x, y we see that any two different points have minimal distance $\frac{1}{b_{max}}$. This gives the last part of the statement of (a). Now, the first part follows directly.

(b) This follows as different points have a minimal distance by (a).

(c) By (a) the points in $B_r(x)$ can be reached from x by paths with not more than $r \cdot b_{max}$ edges. Moreover, in the relevant paths no edge can be longer than r . Thus, we will just estimate the number of path with not more than $r \cdot b_{max}$ edges of length not exceeding r . Now, by (B) and (M) the number $N_r(p)$ of edges going out from an arbitrary $p \in X$ with length not exceeding r is bounded by

$$N_r(p) \cdot \frac{1}{r} \leq \sum_{z \in X} b(p, z) \leq \delta m_{max}.$$

The preceding considerations directly imply the given bound for $\#B_r(x)$. From (M) we then obtain (V).

(d) By (c) any ball has only finitely many points. Consider now arbitrary $x, y \in X$ and set $r := d(x, y)$. Then, y belongs to $B_{r+1}(x)$. By (b) the ball $B_{r+1}(x)$ has only finitely many points. Thus, there exist only finitely many paths in $B_{r+1}(x)$ and every path from x to y with length less than $d(x, y) + 1$ lies completely in $B_{r+1}(x)$. So the infimum over the lengths of all paths between x and y can be calculated by taking the minimum over the lengths of paths between x and y in $B_{r+1}(x)$ and this implies (d).

This easily gives the desired statement. \square

Our **setting for the remaining part of the paper** will be a connected (X, b, m) such that (B) and (M) hold. By the previous proposition, this will imply validity of (F), (G) and (V).

Of course, connectedness is not a real issue: if the graph is not connected it decomposes into connected clusters and the Laplacian will just be the direct sum of the Laplacians on the corresponding clusters. Hence our statements will remain true if properly adapted. The only change is that d as defined above is no longer a metric in the sense that the value infinity might occur.

Although our setting allows for more general weighted graphs readers may always assume that we are dealing with usual combinatorial graphs and the associated Laplacians. Our results are relevant and new in this more specialized setting as well for which we now single out two particularly important classes. Note that the usual euclidean lattices belong to the first class of examples and - up to a multiplication of the measure by a constant - also to the second class of examples.

Example 2.2 (Combinatorial situation). *Starting from a combinatorial graph $G = (X, E)$ we set $b(x, y) = 1$ whenever there is an edge from x to y and*

$b(x, y) = 0$ else and $m = 1$. Then our Laplacian agrees with the usual graph Laplacian and the distance is the wellknown combinatorial or graph distance. Our basic assumptions are satisfied if and only if G is connected and the vertex degree is uniformly bounded.

Example 2.3 (Normalized situation). *Let X be an arbitrary countable set with more than one element and let $b : X \times X \rightarrow [0, \infty)$ be symmetric with $b(x, x) = 0$ and $\sum_y b(x, y) < \infty$ for all $x \in X$. Assume that X is connected and define*

$$m : X \rightarrow [0, \infty), m(x) := \sum_{y \in X} b(x, y).$$

Due to connectedness there must exist from any $x \in X$ a $y \in X$ with $b(x, y) > 0$ and we find $m(x) > 0$ for any $x \in X$. As is clear from the construction the condition (B) holds (with $\delta = 1$). In particular, the form \mathcal{E} and the operator H are automatically bounded in this situation. So, in this case the basic assumption is satisfied if and only if the graph is connected and m is bounded.

Remark 2.4. *The metric d and related metrics are sometimes discussed under the name of path metrics on graphs. They have appeared in various places. A study of topological features of graphs equipped with d is given in [19]. Completeness of the space X equipped with respect to path metrics has played a role in recent investigations of essential selfadjointness of Laplacians on graphs, [43, 37, 24]. An important step in the considerations of [37] gives that completeness with respect to a certain path metric implies finiteness of metric balls. This was generalized in [24] to a Hopf-Rinow type theorem giving that for any path metric completeness of X is equivalent to finiteness of metric balls and implies existence of geodesics. A further discussion of d and other metrics in the context of suitable (pre)compactness conditions for graphs is given in [20]. Our framework given by (B) and (M) and the consequences for the geometry seem not to have been studied before.*

3 Lower bounds for the Dirichlet Laplacian

From the introduction we know that an interesting situation to study is that on $D \subset X$ we have an infinite potential. Denoting by $\Omega := X \setminus D$ we get the form

$$\mathcal{E}_\Omega(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot) \text{ on } \text{dom}(\mathcal{E}_\Omega) = \{f \in \text{dom}(\mathcal{E}) = \ell^2(X, m) \mid f = 0 \text{ on } D\}$$

as the limit in the strong resolvent sense of

$$H + t1_D \text{ as } t \rightarrow \infty,$$

(see Section 4 for further details).

We identify $\ell^2(\Omega, m)$ with $\{f \in \ell^2(X, m) \mid f = 0 \text{ on } D\}$ and get an associated selfadjoint operator H_Ω defined on $\ell^2(\Omega, m)$. For us, \mathcal{E}_Ω and H_Ω will be the restriction of the energy form and the Laplacian, respectively, to Ω with *Dirichlet boundary conditions*.

Remark 3.1. (a) *It is quite reasonable to call H_Ω the Dirichlet Laplacian. In the continuum euclidean case, under mild regularity assumptions on the boundary of Ω , open in \mathbb{R}^N , it holds that*

$$-\Delta_\Omega = \lim_{t \rightarrow \infty} (-\Delta + t1_{\mathbb{R}^N \setminus \Omega})$$

is the Dirichlet Laplacian and the convergence holds in the strong resolvent sense, see Section 4 for references.

(b) *It is also reasonable to call another operator the Dirichlet Laplacian in the discrete case, see for example Section 5.2 in [30]. This is adopted by many authors who add a penalty term in order to force the subadditivity known from the continuum case. In fact our Dirichlet Laplacian does in general not obey that $H_{U \cup V}$ is smaller than $H_U \oplus H_V$ for disjoint U and V .*

In our application to unique continuation the main role is played by D , the set on which the potential barrier is given. For the present section we slightly change the point of view and concentrate on the set $\Omega \subset X$. We define the *inradius* of Ω by

$$\text{Inr}(\Omega) := \sup\{r > 0 \mid \exists x \in \Omega : U_r(x) \subset \Omega\}.$$

We are particularly interested in

$$\lambda_\Omega := \min \sigma(H_\Omega),$$

the bottom of the spectrum of the Dirichlet Laplacian H_Ω . Note that this latter operator is defined on $\ell^2(\Omega)$ but its definition is always to be understood relative to the bigger ambient graph X .

We will first deal with the finite volume situation in the next theorem.

Theorem 3.2. *Let (X, b, m) be as above, in particular connected and such that (B) and (M) hold. Let a non-empty $\Omega \subset X$ with $\Omega \neq X$ be given and assume $\text{vol}(\Omega) < \infty$ and $\text{Inr}(\Omega) < \infty$.*

(a) *We have*

$$\lambda_\Omega \geq \frac{1}{\text{Inr}(\Omega)\text{vol}(\Omega)}.$$

(b) *If $\text{vol}(X) < \infty$, then*

$$\lambda_\Omega \leq \|H\| \frac{\text{vol}(X) - \text{vol}(\Omega)}{\text{vol}(X)}.$$

Proof. Ad (a): Let $R > \text{Inr}(\Omega)$. Let $f \in \text{dom}(\mathcal{E}_\Omega)$ and $x \in \Omega$. By definition of the inradius there is $x_0 \in U_R(x) \setminus \Omega$. In particular, there is a path $\gamma = (x_0, \dots, x_k)$ from x_0 to $x = x_k$ of length at most R and $f(x_0) = 0$. Therefore,

$$\begin{aligned} |f(x)|^2 &= |f(x) - f(x_0)|^2 \\ &= \left| \sum_{j=0}^{k-1} \sqrt{b(x_j, x_{j+1})} (f(x_{j+1}) - f(x_j)) \frac{1}{\sqrt{b(x_j, x_{j+1})}} \right|^2 \\ &\leq \sum_{j=0}^{k-1} b(x_j, x_{j+1}) |f(x_{j+1}) - f(x_j)|^2 \sum_{j=0}^{k-1} \frac{1}{b(x_j, x_{j+1})} \\ &\leq \mathcal{E}_\Omega(f, f) R. \end{aligned}$$

Since $\|f\|^2 \leq \sup_{x \in \Omega} |f(x)|^2 \cdot \text{vol}(\Omega)$ we get

$$\|f\|^2 \leq R \cdot \text{vol}(\Omega) \cdot \mathcal{E}_\Omega(f, f)$$

and this is the desired lower bound, since $R > \text{Inr}(\Omega)$ was arbitrary.

Ad (b): Note that $\mathcal{E}(1, 1) = 0$ under these conditions. We will define a suitable trial function. In fact, let $\phi = c \cdot 1_\Omega$ normalized, so that $c = \text{vol}(\Omega)^{-\frac{1}{2}}$. Since H and consequently H_Ω is bounded, $\phi \in \text{dom}(\mathcal{E}_\Omega)$. We will estimate the energy of ϕ by calculating the projection $\phi_0 = P_0 \phi$, where P_0 is the orthogonal projection onto the constant functions and hence leaves H invariant. We get

$$\phi_0 = \frac{1}{\text{vol}(X)} \langle \phi, 1_X \rangle 1_X \text{ with } \langle \phi, 1_X \rangle = \sum_{x \in \Omega} c m(x) = \text{vol}(\Omega)^{\frac{1}{2}}$$

and therefore

$$\|\phi_0\|^2 = \frac{\text{vol}(\Omega)}{\text{vol}(X)} \text{ and } \|\phi - \phi_0\|^2 = 1 - \|\phi_0\|^2 = \frac{\text{vol}(X) - \text{vol}(\Omega)}{\text{vol}(X)}.$$

Since $H\phi_0 = 0$,

$$\mathcal{E}(\phi, \phi) = \mathcal{E}(\phi - \phi_0, \phi - \phi_0) \leq \|H\| \|\phi - \phi_0\|^2 = \|H\| \frac{\text{vol}(X) - \text{vol}(\Omega)}{\text{vol}(X)}.$$

As ϕ is supported in Ω we have $\mathcal{E}_\Omega(\phi, \phi) = \mathcal{E}(\phi, \phi)$ by the definition of \mathcal{E}_Ω and the preceding estimate gives λ_Ω of H_Ω . \square

Remark 3.3. (a) For finite combinatorial graphs, the lower bound is a familiar bound and our proof follows known lines, compare Lemma 1.9 in [11].

(b) The upper bound is interesting as it shows that a lower bound like in the continuum euclidean case for domains with nice enough boundary, namely in the form $\text{const} \cdot \text{Inr}(\Omega)^{-2}$, see Theorem 1.5.8 in [13], will not be possible! Indeed, for any connected finite graph X we can take D to consist of just a single element of X . Then,

$$\lambda_\Omega \leq \|H\| \frac{\text{vol}(X) - \text{vol}(\Omega)}{\text{vol}(X)} \leq \|H\| \frac{m_{\max}}{\text{vol}(\Omega)}$$

will be bounded in terms of the inverse volume of Ω . Now, this can be much smaller than a second power of the inverse inner radius as can be seen by considering e.g. a ball in an N -dimensional euclidean lattice with $N \geq 3$ and choosing as set D just the center of this ball.

To lift the above result to the case of infinite volume, we introduce the concept of a Voronöi decomposition. This concept may be of interest in other contexts as well. In fact, for combinatorial Laplacians it has already proven useful in [41].

Definition 3.4. Let (X, b, m) be as above and $D \subset X$ non-empty. A Voronöi decomposition of X with centers from D is a pairwise disjoint family $(V_p)_{p \in D}$ such that following conditions hold:

- (V1) For each $p \in D$ the point p belongs to V_p and for all $x \in V_p$ there exists a path γ from p to x that lies in V_p and satisfies $L(\gamma) = d(p, x)$.
- (V2) For each $p \in D$ and for all $x \in V_p$ the inequality $d(p, x) \leq d(q, x)$ holds for any $q \in D$.
- (V3) $\bigcup_{p \in D} V_p = X$.

Remark 3.5. The condition (V1) and (V2) imply that for any $p \in D$

- the set V_p contains p and is connected and
- any $x \in V_p$ satisfies $d(p, x) \leq d(q, x)$ for any $q \in D$.

However, it is not hard to see by examples that (V1) and (V2) are even stronger than these two conditions, i.e. that connectedness of the V_p does not imply that they contain geodesics.

In our investigation of Voronöi decompositions, we will need some further concepts. We define the *covering radius* of D by

$$\text{Covr}(D) := \inf\{R > 0 \mid \bigcup_{p \in D} B_R(p) = X\} \in [0; \infty],$$

with the usual convention $\inf \emptyset = \infty$ and say that D is *relatively dense*, provided $\text{Covr}(D) < \infty$.

Lemma 3.6. *Let (X, b, m) be as above, $D \subset X$ and $\Omega := X \setminus D$. Then*

$$\text{Covr}(D) = \text{Inr}(\Omega).$$

Proof. Let $R < \text{Covr}(D)$ (which is set to ∞ if D is not relatively dense). Then $\bigcup_{p \in D} B_R(p) \neq X$ which means that there is $x_0 \in \Omega$ with $B_R(x_0) \cap D = \emptyset$ and therefore $\text{Inr}(\Omega) \geq R$. Consequently, $\text{Covr}(D) \leq \text{Inr}(\Omega)$.

Conversely, $R < \text{Inr}(\Omega)$ gives $x_0 \in \Omega$ and $R < \tilde{R} < \text{Inr}(\Omega)$ s.t. $B_R(x_0) \subset U_{\tilde{R}}(x_0) \subset \Omega$ which means that $x_0 \notin \bigcup_{p \in D} B_R(p)$ and, therefore, $R < \text{Covr}(D)$. Consequently, $\text{Covr}(D) \geq \text{Inr}(\Omega)$. \square

Remark 3.7. *If D is relatively dense in X then the infimum in the definition of the covering radius is even a minimum i.e. $X = \bigcup_{p \in D} B_R(p)$ for $R = \text{Covr}(D)$ holds. To see this choose an arbitrary $x \in X$ and consider $B_{R+1}(x) \cap D$. By the definition of the covering radius this set contains a sequence $(p_n) \subset D$ with $\inf d(p_n, x) \leq R$. Moreover, by (F) this set is finite. Thus, it must contain a $p \in D$ with $d(p, x) \leq R$. As $x \in X$ was arbitrary the desired statement follows.*

Here is our result on existence of a Voronöi decomposition.

Proposition 3.8. *Let (X, b, m) be as above and assume that $D \subset X$ is non-empty. Then there exists a Voronöi decomposition with centers from D . Moreover, whenever $R = \text{Covr}(D)$ is finite then any Voronöi decomposition $(V_p)_{p \in D}$ of X with centers from D has the property that $V_p \subset B_R(p)$ for all $p \in D$.*

Proof. We first show existence of a Voronöi decomposition with centers from D . A family $(V_p)_{p \in D}$ of pairwise disjoint subsets of X is called admissible, if it satisfies (V1) and (V2) from Definition 3.4 above. Evidently, $V_p = \{p\}$, $p \in D$, gives such an admissible family. With the obvious ordering we can apply Zorn's lemma and get a maximal admissible family. We will show now that such a maximal family is a Voronöi decomposition, i.e., satisfies as well (V3):

$$\bigcup_{p \in D} V_p = X.$$

Assume otherwise. Then there exists an $x \in X$ which does not belong to

$$W := \bigcup_{p \in D} V_p.$$

Now as X is connected and D is not empty, there exists an $R > 0$ such that the set

$$S := B_R(x) \cap D$$

is not empty. Indeed, we may just take $R = d(x, q)$ for any $q \in D$. Moreover, S is finite as $B_R(x)$ is finite (due to Proposition 2.1). Therefore, there exists a $p \in S$ with minimal distance to x i.e. with

$$d(p, x) \leq d(u, x) \quad (3)$$

for any $u \in S$. By $p \in S$, clearly, $d(p, x) \leq R$ holds. Thus, (3) holds also for $u \in (X \setminus B_R(x)) \cap D$. Hence, we see that (3) holds for all $u \in D$.

Moreover, as our space is geodesic due to Proposition 2.1, there exists a path $\gamma = (x_0, \dots, x_k)$ with $x_0 = p$ and $x_k = x$ and

$$d(p, x) = L(\gamma) = \sum_{j=0}^{k-1} b(x_j, x_{j+1}).$$

Then, for any $u \in D$ we must have

$$d(u, x_l) \geq \sum_{j=0}^{l-1} b(x_j, x_{j+1}) \quad (4)$$

for any $l = 1, \dots, k$, as otherwise we would arrive at

$$d(u, x) \leq d(u, x_l) + d(x_l, x) < \sum_{j=0}^{k-1} b(x_j, x_{j+1}) = d(p, x)$$

which contradicts (3). Consider now the smallest index $l \in \{0, \dots, k-1\}$ with $x_l \in W$ and $x_{l+1} \notin W$. (Such an l exists as $x_0 = p \in W$ and $x_k = x \notin W$ by our assumption.) Let $q \in D$ be such that $x_l \in V_q$. By (V2) (applied to V_q) we then have

$$d(q, x_l) \leq d(p, x_l) \leq \sum_{j=0}^{l-1} b(x_j, x_{j+1}).$$

Combined with (4) this gives

$$d(q, x_l) = d(p, x_l) = \sum_{j=0}^{l-1} b(x_j, x_{j+1}). \quad (5)$$

Putting this together we arrive at

$$\begin{aligned} d(q, x_{l+1}) &\leq d(q, x_l) + d(x_l, x_{l+1}) \\ &\leq d(q, x_l) + b(x_l, x_{l+1}) \\ (5) &= \sum_{j=0}^l b(x_j, x_{j+1}) \\ (4) &\leq d(u, x_{l+1}) \end{aligned}$$

for any $u \in D$. This chain of inequalities gives not only

$$d(q, x_{l+1}) \leq d(u, x_{l+1})$$

for all $u \in D$ but also (if we set $u = q$)

$$d(q, x_{l+1}) = d(q, x_l) + b(x_l, x_{l+1}).$$

Thus, we could add x_{l+1} to V_q and obtain the admissible decomposition $(\tilde{V}_u)_{u \in D}$ with $\tilde{V}_q := V_q \cup \{x_{l+1}\}$ and $\tilde{V}_u = V_u$ for $q \neq u \in D$. This is a contradiction to maximality. Thus, we infer that a maximal admissible family satisfies (V1), (V2) and (V3).

We now show the last statement. So, let (V_p) , $p \in D$, be a Voronöi decomposition. Let $p \in D$ and $x \in V_p$ be arbitrary. As $\text{Covr}(D) = R < \infty$ there must exist a $q \in D$ with $d(q, x) \leq R$. By (V2), we then infer

$$d(p, x) \leq d(q, x) \leq R.$$

This shows $V_p \subset B_R(p)$. □

Remark 3.9. *We note that the proof of the previous proposition does not require (B) and (M) but only the weaker Heine-Borel property (F) (and the resulting existence of geodesics (G)). Thus, the proposition will be true in even more general situations than the standard setting of our paper. Note also that the existence statement of the proposition does not need relative denseness of the set D .*

Theorem 3.10. *Let (X, b, m) be as above, in particular connected and such that (B) and (M) hold and assume that $D \subset X$ is relatively dense, $\Omega := X \setminus D$. Then,*

$$\lambda_\Omega \geq \frac{1}{\text{Inr}(\Omega) \cdot \text{vol}(\text{Inr}(\Omega))},$$

where

$$\text{vol}(s) := \sup_{x \in X} \text{vol}(B_s(x)).$$

Remark 3.11. (a) *Note that $\text{vol}(s) < \infty$ for any $s \geq 0$ due to Proposition 2.1.*

(b) *The proof shows that we can actually replace $\text{vol}(s)$ by the slightly better $\text{vol}_\Omega(s) := \sup_{x \in X} \text{vol}(B_s(x) \cap \Omega)$.*

Proof. Let V_p , $p \in D$, be the Voronöi decomposition from the preceding proposition. Then, we have

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{x, y} b(x, y) (f(x) - f(y))^2 \geq \frac{1}{2} \sum_{p \in D} \sum_{x, y \in V_p} b(x, y) (f(x) - f(y))^2.$$

To each sum over V_p we now apply Theorem 3.2 with $D = \{p\}$ and $\Omega = V_p \setminus \{p\}$ to obtain

$$\mathcal{E}(f, f) \geq \frac{1}{2} \sum_{p \in D} \frac{1}{\text{Inr}(V_p \setminus \{p\}) \text{vol}(V_p \setminus \{p\})} \sum_{x \in V_p} |f(x)|^2 m(x).$$

By the previous proposition we know $V_p \subset B_R(p)$ with $R = \text{Covr}(D)$. Moreover, by (V1) any $x \in V_p$ is connected to p by a path in V_p of length $d(p, x) \leq R$. This gives $\text{Inr}(V_p \setminus \{p\}) \leq R$. As $R = \text{Covr}(D) = \text{Inr}(\Omega)$ (see Lemma 3.6) the desired statement follows easily. \square

4 The large coupling limit

In this section we study the large coupling limit

$$H + t1_D \rightarrow H_\Omega$$

more thoroughly. Recall that in the situation we have in mind, H is a weighted Laplacian on some weighted $\ell^2(X, m)$ and $D \subset X$ is a subset, $\Omega = X \setminus D$. We already noticed that the above convergence takes place in the strong resolvent sense, as can easily be seen from Kato's monotone convergence theorem, see [26], Thm 3.13a, p.461 for the densely defined and [42], Thm. 4.1, p. 383 for the general case. Under the assumption that H is bounded, we actually see that we even have norm resolvent convergence with optimal decay rate $1/t$ and uniform bounds that depend on $\|H\|$ only, see Proposition 4.2 below.

The analogous problem is much more intricate in the continuum. There, with $H = -\Delta$, one needs certain regularity assumptions on Ω to even get strong resolvent convergence and these regularity assumptions will not suffice to decide norm resolvent convergence. In this context we refer to [6, 7] for results on norm convergence in a rather general framework and further references.

In our case, we don't even need to take into account the special structure, so in what follows, let \mathcal{H} be a Hilbert space, $P : \mathcal{H} \rightarrow \mathcal{H}_1$ the orthogonal projection onto a closed subspace $\{0\} \neq \mathcal{H}_1 \subsetneq \mathcal{H}$ (to avoid trivialities), Q the orthogonal projection onto $\mathcal{H}_1^\perp =: \mathcal{H}_2$, $0 \leq H$ a bounded selfadjoint operator on \mathcal{H} and

$$H_t := H + tQ^*Q.$$

Again, monotone convergence implies that the corresponding forms \mathcal{E}_t converge, as $t \rightarrow \infty$, to the closed form \mathcal{E}_∞ given by

$$\text{dom}(\mathcal{E}_\infty) = \{f \in \mathcal{H} \mid \sup_t \langle H_t f, f \rangle < \infty\} = \mathcal{H}_1$$

$$\mathcal{E}_\infty(f, g) = \langle Hf, g \rangle$$

Remark 4.1. The unique selfadjoint operator in \mathcal{H}_1 associated with \mathcal{E}_∞ is given by PHP^* .

Proposition 4.2. With $c = c(\|H\|)$ we get

$$\|(H_t + 1)^{-1} - P^*(PHP^* + 1)^{-1}P\| \leq \frac{4\|H + 1\|^2}{1 + t}.$$

for $t \geq 2\|H + 1\|^2$.

Note that $P^*(PHP^* + 1)^{-1}P = (PHP^* + 1)^{-1} \oplus 0$, the resolvent of PHP^* which is defined on \mathcal{H}_1 , extended by 0 to $\mathcal{H}_1^\perp = \mathcal{H}_2$.

Proof of the Proposition. We use the Schur complement by decomposing H_t according to $\mathcal{H}_1 \oplus \mathcal{H}_2$ into a block operator matrix

$$H_t = \begin{pmatrix} PH_tP^* & PH_tQ^* \\ QH_tP^* & QH_tQ^* \end{pmatrix}.$$

Since $PQ^* = QP^* = 0$,

$$H_t + 1 = \begin{pmatrix} P(H + 1)P^* & P(H + 1)Q^* \\ Q(H + 1)P^* & Q(H + 1)Q^* \end{pmatrix} =: \begin{pmatrix} A & B \\ B^* & D_t \end{pmatrix}.$$

For $t \geq 0$, D_t is invertible with $\|D_t^{-1}\| \leq (1 + t)^{-1}$. Consequently, the Schur complement

$$S_t := P(H + 1)P^* - BD_t^{-1}B^*$$

is boundedly invertible for large enough t , since

$$P(H + 1)P^* \geq 1 \quad \text{and} \quad \|BD_t^{-1}B^*\| \leq \|H + 1\|^2 \frac{1}{1 + t}.$$

More precisely, we get

$$S_t \geq \frac{1}{2} \quad \text{and} \quad \|S_t^{-1}\| \leq 2 \quad \text{for } t \geq 2\|H + 1\|^2.$$

Using the Schur complement to invert $H_t + 1$ gives

$$(H_t + 1)^{-1} = \begin{pmatrix} S_t^{-1} & -S_t^{-1}BD_t^{-1} \\ -D_t^{-1}B^*S_t^{-1} & D_t^{-1}(1 + B^*S_t^{-1}B)D_t^{-1} \end{pmatrix},$$

Therefore,

$$\begin{aligned} & (H_t + 1)^{-1} - P^*(PHP^* + 1)^{-1}P \\ &= \begin{pmatrix} S_t^{-1} - (P(H + 1)P^*)^{-1} & -S_t^{-1}BD_t^{-1} \\ -D_t^{-1}B^*S_t^{-1} & D_t^{-1}(1 + B^*S_t^{-1}B)D_t^{-1} \end{pmatrix} \end{aligned}$$

can be bounded in norm by

$$2 \max\{\|S_t^{-1} - (P(H + 1)P^*)^{-1}\|, \|D_t^{-1}B^*S_t^{-1}\|, \|D_t^{-1}(1 + B^*S_t^{-1}B)D_t^{-1}\|\}.$$

Using the resolvent equation for the first term as well as the above bounds gives the claim. \square

Lemma 4.3. *Let $H_2 \geq 0$ be a selfadjoint operator on \mathcal{H} , $H_1 \geq 0$ a selfadjoint operator (possibly on a subspace \mathcal{H}_1), $\lambda_i := \min \sigma(H_i)$ for $i = 1, 2$ and assume that $\lambda_1 \geq \lambda_2$. Then*

$$\begin{aligned} 0 \leq \lambda_1 - \lambda_2 &\leq (\lambda_1 + 1)^2 \|(H_1 + 1)^{-1} - (H_2 + 1)^{-1}\| \\ &\leq \|H_1 + 1\|^2 \|(H_1 + 1)^{-1} - (H_2 + 1)^{-1}\|, \end{aligned}$$

the latter provided H_1 is bounded.

Proof. Denote $\delta := \|(H_1 + 1)^{-1} - (H_2 + 1)^{-1}\|$. Since $H_i \geq \lambda_i$ we get $(H_i + 1)^{-1} \leq (\lambda_i + 1)^{-1}$ which gives $(H_2 + 1)^{-1} \leq (\lambda_1 + 1)^{-1} + \delta$. Thus

$$H_2 + 1 \geq \frac{1}{\frac{1}{\lambda_1 + 1} + \delta} = \frac{\lambda_1 + 1}{1 + \delta(\lambda_1 + 1)}$$

and

$$\lambda_2 \geq \frac{\lambda_1 + 1 - 1 - \delta(\lambda_1 + 1)}{1 + \delta(\lambda_1 + 1)} = \lambda_1 - \delta \left(\frac{(\lambda_1 + 1)^2}{1 + \delta(\lambda_1 + 1)} \right) \geq \lambda_1 - (\lambda_1 + 1)^2 \delta,$$

as claimed. In case that H_1 is bounded, the spectrum is bounded by the norm. This argument extends to the case where H_1 is defined on a subspace \mathcal{H}_1 , with $(H_1 + 1)^{-1}$ is to be read as $(H_1 + 1)^{-1} \oplus 0$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$. \square

Given the previous two results we immediately infer the following.

Corollary 4.4. *In the situation of Proposition 4.2, let*

$$\lambda_t := \min \sigma(H_t) \text{ and } \lambda_\infty := \min \sigma(PHP^*).$$

Then, we have $\lambda_\infty \geq \lambda_t$ for all $t \geq 0$ and for $t \geq 2\|H + 1\|^2$,

$$\lambda_t \geq \lambda_\infty - \frac{4\|H + 1\|^2(\lambda_\infty + 1)^2}{t + 1} \geq \lambda_\infty - \frac{4\|H + 1\|^4}{t + 1}.$$

5 A quantitative unique continuation result for the Laplacian on graphs

We are now in position to derive our main result by combining what we have established so far. We let (X, b, m) be as above, in particular connected and such that (B) and (M) hold and assume that $D \subset X$ is relatively dense, $\Omega := X \setminus D$.

We recall that by [10], Thm 1.1,

$$P_I(H)1_D P_I(H) \geq \kappa P_I(H)$$

provided there is $t > 0$ such that

$$\max I < \min \sigma(H + t1_D) =: \lambda_t \tag{6}$$

Actually, in this case

$$\kappa \geq \frac{\lambda_t - \max I}{t}.$$

Moreover, in virtue of Theorem 3.10 we have a lower bound on $\lambda_\Omega = \min \sigma(H_\Omega)$ viz

$$\lambda_\Omega \geq \frac{1}{\text{Inr}(\Omega) \cdot \text{vol}(\text{Inr}(\Omega))}$$

and, by Corollary 4.4 applied with $\lambda_\infty = \lambda_\Omega$, we have a lower bound

$$\lambda_t \geq \lambda_\Omega - 4\|H + 1\|^2(\lambda_\Omega + 1)^2 \frac{1}{t+1} \quad (7)$$

for $t \geq 2\|H + 1\|^2$. From these bounds we easily get

Theorem 5.1. *Let (X, b, m) be as above, $D \subset X$ relatively dense and $\Omega := X \setminus D \neq \emptyset$. Let $I \subset \mathbb{R}$ such that $\max I < \lambda_\Omega$. Then*

$$P_I(H)1_D P_I(H) \geq \frac{(\lambda_\Omega - \max I)^2}{16\|H + 1\|^2(\lambda_\Omega + 1)^2} P_I(H). \quad (8)$$

Proof. Using (7) we find

$$P_I(H)1_D P_I(H) \geq \kappa P_I(H)$$

for a κ satisfying

$$\kappa \geq \sup_{t \geq 2\|H+1\|^2} \frac{\lambda_\Omega - \max I - 4\|H + 1\|^2(\lambda_\Omega + 1)^2 \frac{1}{t+1}}{t}.$$

To make things easier we replace $\frac{1}{t+1}$ by $\frac{1}{t}$ which gives a lower bound; hence, we are left to find the maximum of

$$f(t) := \frac{c_0}{t} - \frac{c_1}{t^2} \text{ for } t \geq 2\|H + 1\|^2$$

for an appropriate choice of c_0 and c_1 . The corresponding argument is

$$t_{max} = 2\frac{c_1}{c_0} = 8\frac{\|H + 1\|^2(\lambda_\Omega + 1)^2}{\lambda_\Omega - \max I} \geq 2\|H + 1\|^2$$

with maximal value

$$\frac{(\lambda_\Omega - \max I)^2}{16\|H + 1\|^2(\lambda_\Omega + 1)^2},$$

the assertion. \square

Given the preceding theorem we can now use the lower bound from Theorem 3.10 and the trivial bound $\lambda_\Omega \leq \|H_\Omega\| \leq \|H\|$ to obtain the following corollary.

Corollary 5.2. *Let (X, b, m) be as above, $D \subset X$ relatively dense and $\Omega := X \setminus D \neq \emptyset$. Let $I \subset \mathbb{R}$ such that $\max I < \frac{1}{\text{Inr}(\Omega) \cdot \text{vol}(\text{Inr}(\Omega))}$. Then*

$$P_I(H)1_D P_I(H) \geq \frac{\left(\frac{1}{\text{Inr}(\Omega) \cdot \text{vol}(\text{Inr}(\Omega))} - \max I\right)^2}{16\|H + 1\|^4} P_I(H). \quad (9)$$

We go on to compare our results with those obtained by Rojas-Molina [39] and Elgart and Klein [16] who treat the euclidean lattices \mathbb{Z}^d . First note that distances are measured somewhat differently in comparison to what we do here: E.g., an R -Delone set in the sense of [39] would have covering radius bounded above by $d \cdot R$. Therefore, we suppress such numerical constants right now to make things easier, and use \gtrsim to indicate the corresponding relations.

The energy range for the uncertainty principle (in our notation the largest possible $\max I$) is given by $\tilde{E}_W \gtrsim R^{-2d-2}$ in [39] while our result gives the better $\lambda_\Omega \gtrsim R^{-d-1}$. In comparison, [16], provide a lower bound $\lambda_\Omega \gtrsim R^{-2d}$ using a Cheeger inequality. Here, again our estimate is better. We discuss this in more detail in the next section.

6 A closer look at the combinatorial situation

In this section we consider the combinatorial situation and exhibit a large class of models, viz combinatorial graphs with subexponential growth of balls, to which our results can be applied. Along the way we also compare our approach to the approach via Cheeger inequalities used in [16].

Throughout this section we consider the case of combinatorial graphs, i.e. a connected graph (X, b, m) with $m \equiv 1$ and b taking values in $\{0, 1\}$ such that with a suitable $\delta \geq 0$ we have $\sum_{y \in X} b(x, y) \leq \delta$ for all $x \in D$. In fact, in this case $\delta := \sup_x \#\{y \in X : b(x, y) = 1\}$ is the maximal vertex degree and

$$\text{vol}(s) = \sup_{x \in X} \text{vol}(B_s(x)) = \sup_{x \in X} \#B_s(x).$$

We will be particularly interested in the case $\inf \sigma(H) = 0$, as we will have non-trivial applications of our main results in this case. As is well known, this case can be characterized via the *Cheeger constant* or *isoperimetric constant*

$$\beta := \inf_{\emptyset \neq S \subset X, \#S < \infty} \frac{\#\partial S}{\text{vol}(S)},$$

where the *combinatorial boundary* ∂S of S is given by

$$\partial S := \{(x, y) \in X \times X : x \in S, y \notin S \text{ and } b(x, y) = 1\}.$$

Indeed, this characterization is given as

$$\inf \sigma(H) = 0 \iff \beta = 0.$$

Here, the implication ' \Leftarrow ' follows easily by a direct computation. Specifically, for any finite set S we find

$$\mathcal{E}(1_S, 1_S) \leq \#\partial S$$

as well as $\|1_S\|^2 = \text{vol}(S)$, where 1_S denotes the characteristic function. The implication ' \Rightarrow ' follows from the Cheeger inequality

$$\inf \sigma(H) \geq \frac{\beta^2}{2\delta}.$$

This inequality goes back to [14] (in a slightly different formulation), see [5] and the discussion below as well. A well-known consequence of these considerations is that $\inf \sigma(H) = 0$ whenever the volume growth of balls is subexponential:

Proposition 6.1. *Assume that for one (and thus all) $x \in X$*

$$\limsup_{n \rightarrow \infty} \frac{\log \text{vol}(B_n(x))}{n} = 0. \quad (10)$$

Then, $\inf \sigma(H) = 0$.

Proof. The assumption implies that

$$\inf_n \frac{\text{vol}(B_{n+1}(x) \setminus B_n(x))}{\text{vol}(B_n(x))} = 0$$

(as otherwise we had $\text{vol}(B_{n+1}(x)) \geq (1 + \alpha)^n \text{vol}(B_1(x))$ with α being the non-vanishing value of the infimum and this would lead to exponential volume growth). Moreover, a direct combinatorial argument shows that

$$\#\partial B_n(x) \leq \delta \cdot \text{vol}(B_{n+1}(x) \setminus B_n(x)).$$

Putting this together we infer $\beta = 0$. Hence, by the preceding considerations the statement on the infimum of the spectrum follows. \square

Now let $D \subset X$ be relatively dense and set $\Omega := X \setminus D \neq \emptyset$. If $\beta = 0$ (or, equivalently, $\inf \sigma(H) = 0$), we obtain from Theorem 3.10

$$\lambda_\Omega = \inf \sigma(H_\Omega) \geq \frac{1}{\text{Inr}(\Omega) \cdot \text{vol}(\text{Inr}(\Omega))} > 0 = \inf \sigma(H).$$

So, the infimum of the spectrum of H_Ω is indeed bigger than the infimum of the spectrum of H . In this case, whenever $I \subset \mathbb{R}$ is an interval containing 0 with $\max I < \frac{1}{\text{Inr}(\Omega) \cdot \text{vol}(\text{Inr}(\Omega))}$, we have $P_I(H) \neq 0$. So, we can in particular apply from Corollary 5.2 to obtain a non-trivial inequality. As $\beta = 0$ holds whenever (10) is satisfied, these considerations provide a large class of examples in which our approach can be carried out.

If $\beta = 0$ and $D \subset X$ is relatively dense it is also possible to obtain a lower bound for $\inf \sigma(H_\Omega)$ via a Cheeger type inequality. Specifically, we define

$$\beta_\Omega := \inf_{\emptyset \neq S \subset \Omega, \#S < \infty} \frac{\#\partial S}{\text{vol}(S)}.$$

For the case of Euclidean lattices this is carried out in [16]. The general case can be inferred from [14, 5], as we will explain below. To apply this in a meaningful way we need an explicit estimate on β_Ω . The case $X = \mathbb{Z}^d$ and $D \subset X$ relatively dense is treated in [16] and it is shown that $\beta_\Omega \geq C/R^d$ with a suitable constant C and R being the covering radius of D . It turns out that a similar bound can be obtained in the general case as well. In fact, this can be shown rather directly based on the Voronöi decomposition provided in Proposition 3.8.

Proposition 6.2. *Let $D \subset X$ be relatively dense with covering radius R and set $\Omega = X \setminus D$. Then,*

$$\beta_\Omega \geq \frac{1}{\text{vol}(R)}.$$

Proof. From Proposition 3.8 we obtain a Voronöi decomposition $(V_p)_{p \in D}$ with centers in D . Let $S \subset \Omega$ be an arbitrary non-empty finite subset of Ω . Consider now an arbitrary $p \in D$ with $V_p \cap S \neq \emptyset$. Then, V_p must contain $u \in X \setminus S$ and a $w \in S$ with $b(u, w) = 1$. (To see this it suffices to consider a path in V_p from a $q \in S \cap V_p \neq \emptyset$ to $p \in D \subset X \setminus S$. Such a path exists by (V1). As such a path starts in S and finishes in the complement of S , there must exist a first edge, where it leaves S . This edge gives the desired points u, w .) So, any V_p that intersects S provides at least one 'boundary edge'. Consequently, with

$$N(S) := \#\{p \in D : V_p \cap S \neq \emptyset\}$$

we have

$$\#\partial S \geq N(S).$$

At the same time we also clearly have

$$\text{vol}(S) \leq N(S) \sup_{p \in D} \text{vol}(V_p) \leq N(S) \text{vol}(R),$$

where we use that any V_p is contained in a ball of radius R . Putting the last two estimates together we find $\#\partial S / \text{vol}(S) \geq \frac{1}{\text{vol}(R)}$. As S was an arbitrary non-empty finite subset of X the desired estimate on β follows. \square

Based on this proposition and the Cheeger inequality from [5] we obtain for $D \subset X$ with $R = \text{Covr}(D)$ the lower bound

$$\lambda_\Omega \geq \frac{\beta_\Omega^2}{2\delta} \geq \frac{1}{2\delta \cdot \text{vol}(R)^2}. \quad (11)$$

In fact, in [5], the main point is to deal with a different isoperimetric constant α , defined in terms of an intrinsic metric ρ that allows for a Cheeger inequality in the case of unbounded vertex degree. In our simpler situation, we can choose as an intrinsic metric the following multiple of the combinatorial metric, namely

$$\rho(\cdot, \cdot) := \delta^{-\frac{1}{2}} d(\cdot, \cdot)$$

and arrive at (11) above by applying Lemma 3.5 from [5]. Note that this is the bound that can also be found in [14], where however, Laplacians are defined in a slightly different way.

Clearly, the lower bound

$$\lambda_\Omega \geq \frac{1}{R \cdot \text{vol}(R)},$$

from Theorem 3.10 is stronger whenever the volume $\text{vol}(R)$ grows faster than linear.

7 Including a potential

In this section we discuss how the ideas presented above allow one in certain cases to include a potential as well.

As usual we assume that we are given a connected graph (X, b, m) satisfying (B) and (M). We denote the set of functions $f : X \rightarrow \mathbb{C}$ which vanish outside a finite set by $C_c(X)$. Let now additionally be given a bounded function

$$V : X \rightarrow \mathbb{R}.$$

Then, we define the form $\mathcal{E}_V := \mathcal{E} + V$ and denote the associated selfadjoint operator by L_V and set

$$\lambda_V := \inf \sigma(L_V).$$

As V is bounded, so are \mathcal{E}_V and L_V . In fact, L_V acts via

$$(L_V f)(x) = \frac{1}{m(x)} \left(\sum_{y \in X} b(x, y)(f(x) - f(y)) + V(x)f(x) \right).$$

By general principles, sometimes discussed under the name of Allegretto-Piepenbrinck theorem, e.g. [21], there exists a non-negative *ground state* to L_V i.e. a function

$$\phi : X \rightarrow (0, \infty)$$

satisfying the summability condition

$$\sum_{y \in X} b(x, y)\phi(y) < \infty$$

for every $x \in X$ as well as

$$\frac{1}{m(x)} \left(\sum_{y \in X} b(x, y)(\phi(x) - \phi(y)) + V(x)\phi(x) \right) - \lambda_V \phi(x) \geq 0$$

for all $x \in X$.¹ Note that all sums in the previous inequality are absolutely convergent due to the summability condition satisfied by ϕ . Then, a *ground state transform* as discussed e.g. in [21], gives

$$(\mathcal{E}_V(f, f) - \lambda_V(f, f)) \geq \mathcal{E}_\phi \left(\frac{f}{\phi}, \frac{f}{\phi} \right) \quad (12)$$

for all $f \in C_c(X)$. Here, \mathcal{E}_ϕ is the form associated to the graph (X, b_ϕ, m_ϕ) with

$$b_\phi(x, y) = \phi(x)\phi(y)b(x, y)$$

for all $x, y \in X$ and $m_\phi(x) = \phi(x)^2 m(x)$ for $x \in X$. Specifically,

$$\mathcal{E}_\phi(f, g) = \frac{1}{2} \sum_{x, y \in X} b_\phi(x, y)(f(x) - f(y))(\overline{g(x)} - \overline{g(y)})$$

for $g \in C_c(X)$.

Assume now that the ground state ϕ is *regular* i.e. there exists a $c \geq 1$, called the *bound* on ϕ with

$$0 < \frac{1}{c} \leq \phi(x) \leq c$$

for all $x \in X$ (see [17] for further discussion of regular ground states). Then, the graph (X, b_ϕ, m_ϕ) satisfies the assumption (B) and (M).

Moreover, the distance d_ϕ associated to (X, b_ϕ, m_ϕ) is equivalent to the distance d associated to (X, b, m) in the sense that we have

$$\frac{1}{c^2} d(x, y) \leq d_\phi(x, y) \leq c^2 d(x, y)$$

for all $x, y \in X$. Similarly, volumes are equivalent for any finite set $S \subset X$ in the sense that we have

$$\frac{1}{c^2} m(S) \leq m_\phi(S) \leq c^2 \text{vol}(S).$$

Let now $D \subset X$ be relatively dense with respect to d with covering radius R . Set $\Omega := X \setminus D$. As d and d_ϕ are equivalent, then D is relatively dense with covering radius less than $c^2 R$ with respect to d_ϕ .

¹If the graph is locally finite i.e. $\#\{y \in X : b(x, y) > 0\} < \infty$ holds for all $x \in X$, one can find ϕ with equality in the previous inequality.

We can now apply Theorem 3.10 to (X, b_ϕ, m_ϕ) . Taking into account the equivalence of metrics and volumes we obtain from this theorem

$$\mathcal{E}_\phi(g, g) \geq \frac{1}{c^4 \cdot R \cdot \text{vol}(c^2 R)} \|g\|_{\ell^2(X, m_\phi)}^2 \quad (13)$$

for all $g \in C_c(\Omega)$. We can then combine (12) and (13) to obtain

$$(\mathcal{E}_V(f, f) - \lambda_V(f, f)) \geq \frac{1}{c^4 \cdot R \cdot \text{vol}(c^2 R)} \|f/\phi\|_{\ell^2(X, m_\phi)}^2 = \frac{1}{c^4 \cdot R \cdot \text{vol}(c^2 R)} \|f\|^2$$

for all $f \in C_c(\Omega)$. So, if we define $\mathcal{E}_{V, \Omega}$ as the restriction of \mathcal{E}_V to $\ell^2(\Omega, m)$ we can summarize the preceding considerations in the following theorem.

Theorem 7.1. *Let (X, b, m) be as above and $V : X \rightarrow \mathbb{R}$ bounded and \mathcal{E}_V the associated form. Let $D \subset X$ be relatively dense with covering radius R . If there exists a regular ground state with bound c to \mathcal{E}_V then,*

$$\mathcal{E}_{V, \Omega} \geq \lambda_V + \frac{1}{c^4 \cdot R \cdot \text{vol}(c^2 R)}$$

holds.

Remark 7.2. *If the metric d satisfies the volume doubling property that there exists an $N > 0$ with $\text{vol}(\alpha s) \leq \alpha^N \text{vol}(s)$ for all $\alpha \geq 1$ and $s > 0$ we can further estimate the bound in the previous theorem as*

$$\mathcal{E}_{V, \Omega} \geq \lambda_V + \frac{1}{c^{4+2N} \cdot R \cdot \text{vol}(R)}.$$

References

- [1] S. Agmon, *Lower bounds for solutions of Schrödinger equations*, J. Analyse Math. **23** (1970), 1–25.
- [2] W. O. Amrein, A.-M. Berthier and V. Georgescu, *L^p -inequalities for the Laplacian and unique continuation*, Ann. Inst. Fourier (Grenoble) **31** (1981), 153–168.
- [3] M. Arai, T. Tokihiro, T. Fujiwara, *Strictly localized states on a two-dimensional Penrose lattice*, Phys. Rev. B **38** (1988), 1621–1626.
- [4] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pure Appl. **36** (1957), 235–249.
- [5] F. Bauer, M. Keller and R. K. Wojciechowski, *Cheeger inequalities for unbounded graph Laplacians*, J. Eur. Math. Soc. **17** (2015), 259–271.
- [6] A. Ben Amor and J. Brasche, *Sharp estimates for large coupling convergence with applications to Dirichlet operators*, J. Funct. Anal. **254** (2008), 454–475.

- [7] J. Brasche and M. Demuth, *Dynkin's formula and large coupling convergence*, J. Funct. Anal. **219** (2005), 34–69.
- [8] D. Borisov, M. Tautenhahn and I. Veselić, *Scale-free quantitative unique continuation and equidistribution estimates for solutions of elliptic differential equations*, arXiv preprint arXiv:1512.06347 (2015)
- [9] J. Bourgain and C. Kenig, *On localization in the continuous Anderson-Bernoulli model in higher dimension*, Invent. Math. **161** (2005), 389–426.
- [10] A. Boutet de Monvel, D. Lenz and P. Stollmann, *An uncertainty principle, Wegner estimates and localization near fluctuation boundaries*, Math. Z. **269** (2011), 663–670.
- [11] F. R. K. Chung, *Spectral graph theory*, CBMS Regional Conference Series in Mathematics, **92**, American Mathematical Society, Providence, RI, 1997.
- [12] Y. Colin de Verdière, F. Truc and N. Torki-Hamza, *Essential self-adjointness for combinatorial Schrödinger operators II - metrically non complete graphs*, Math. Phys. Anal. Geom. **14** (2011), 21–38.
- [13] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics **92**, Cambridge University Press, Cambridge, 1989.
- [14] J. Dodziuk, *Difference Equations, Isoperimetric Inequality and Transience of Certain Random Walks*, Trans. Amer. Math. Soc. **284** (1984), 787–794.
- [15] J. Dodziuk, P. Linnell, V. Mathai, T. Schick and S. Yates, *Approximating L^2 -invariants, and the Atiyah conjecture*, Commun. Pure Appl. Math. **56** (2003), 839–873.
- [16] A. Elgart and A. Klein, *Ground state energy of trimmed discrete Schrödinger operators and localization for trimmed Anderson models*, J. Spectr. Theory **4** (2014), 391–413.
- [17] R. Frank, B. Simon and T. Weidl, *Eigenvalue bounds for perturbations of Schrödinger operators and Jacobi matrices with regular ground states*, Comm. Math. Phys. **282** (2008), 199–208.
- [18] T. Fujiwara, M. Arai, T. Tokihiro and M. Kohmoto, *Localized states and self-similar states of electrons on a two-dimensional Penrose lattice*, Phys. Rev. B **37** (1988), 2797–2804
- [19] A. Georgakopoulos, *Graph topologies induced by edge lengths*, Discrete Math. **311** (2011), 1523–1542.
- [20] A. Georgakopoulos, S. Haeseler, M. Keller, D. Lenz and R. Wojciechowski, *Graphs of finite measure*, J. Math. Pures Appl. (9) **103** (2015), 1093–1131.
- [21] S. Haeseler and M. Keller, *Generalized solutions and spectrum for Dirichlet forms on graphs*, Random walks, boundaries and spectra, 181–199, Progr. Probab. **64**, Birkhäuser/Springer, Basel, 2011
- [22] S. Haeseler, M. Keller, D. Lenz and R. K. Wojciechowski, *Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions*, J. Spectr. Theory, **2** (2012), 397–432.

- [23] L. Hörmander, *Uniqueness theorems for second order elliptic partial differential equations*, Comm. Part. Diff. Equations **8** (1983), 21–64.
- [24] X. Huang, M. Keller, J. Masamune and R. Wojciechowski, *A note on self-adjoint extensions of the Laplacian on weighted graphs*, J. Funct. Anal. **265** (2013), 1556–1578.
- [25] D. Jerison and C. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math. **121** (1985), 463–488.
- [26] T. Kato, *Perturbation theory for linear operators*. Reprint of the 1980 edition, Classics in Mathematics, Springer, Berlin, 1995.
- [27] M. Keller, *Curvature, geometry and spectral properties of planar graphs*, Discrete Comput. Geom. **46** (2011), 500–525.
- [28] M. Keller, *Intrinsic metrics on graphs: a survey*, Mathematical technology of networks, 81–119, Springer Proc. Math. Stat. **128**, Springer, Cham, 2015
- [29] M. Keller and D. Lenz, *Dirichlet forms and stochastic completeness of graphs and subgraphs*, J. Reine Angew. Math. **666** (2012), 189–223.
- [30] W. Kirsch, *An invitation to random Schrödinger operators. With an appendix by F. Klopp.*, Panor. Synthèses **25**, Random Schrödinger operators, 1–119, Soc. Math. France, Paris, 2008
- [31] S. Klassert, D. Lenz, N. Peyerimhoff and P. Stollmann, *Elliptic operators on planar graphs: unique continuation for eigenfunctions and nonpositive curvature*, Proc. Amer. Math. Soc. **134** (2006), 1549–1559.
- [32] S. Klassert, D. Lenz and P. Stollmann, *Discontinuities of the integrated density of states for random operators on Delone sets*, Comm. Math. Phys. **241** (2003), 235–243.
- [33] A. Klein, *Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for non-ergodic random Schrödinger operators*, Comm. Math. Phys. **323** (2013), 1229–1246.
- [34] A. Klein and C. Tsang, *Quantitative unique continuation principle for Schrödinger operators with singular potentials*, Proceedings of the American Mathematical Society **144** (2016), 665–679.
- [35] M. Kohmoto and B. Sutherland, *Electronic States on a Penrose Lattice*, Phys. Rev. Lett. **56** (1986), 2740–2743.
- [36] M. Krajić and T. Fujiwara, *Strictly localized eigenstates on a three-dimensional Penrose lattice*, Phys. Rev. B **38** (1988), 12903–12907.
- [37] O. Milatovic, *Essential self-adjointness of magnetic Schrödinger operators on locally finite graphs*, Integral Equations Operator Theory **71** (2011), 13–27.
- [38] I. Nakić, M. Täufer, M. Tautenhahn and I. Veselić, *Scale-free uncertainty principles and Wegner estimates for random breather potentials*, Comptes Rendus Mathématique **353** (2015), 919–923.
- [39] C. Rojas-Molina, *The Anderson model with missing sites*, Oper. Matrices **8** (2014), 287–299.

- [40] C. Rojas-Molina and I. Veselić, *Scale-free unique continuation estimates and applications to random Schrödinger operators*, Comm. Math. Phys. **320** (2013), 245–274.
- [41] R. Samavat, P. Stollmann and I. Veselić, *Lifshitz asymptotics for percolation Hamiltonians*, Bull. Lond. Math. Soc. **46** (2014), 1113–1125.
- [42] B. Simon, *A canonical decomposition for quadratic forms with applications to monotone convergence theorems*, J. Funct. Anal. **28** (1978), 377–385.
- [43] N. Torki-Hamza, *Laplaciens de graphes infinis métriquement complets*, Confluentes Math. **2** (2010), 333–350.
- [44] I. Veselić, *Spectral analysis of percolation Hamiltonians*, Math. Ann. **331** (2005), 841–865.